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ENTHALPY FLOW TRANSITION LOSSES IN REGENERATIVE CRYOCOOLERS

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ABSTRACT

In regenerative cryocoolers, enthalpy flow is characterized by an oscillating temperature within the working fluid. When there are changes in enthalpy flow, the amplitude of the temperature oscillation also changes. This is an irreversible process that generates entropy. Thus, this process is a loss mechanism. Such losses occur at the transition between heat exchangers (isothermal regions) and adiabatic regions.

This paper presents a generalized method of calculating these losses. For a sinusoidal temperature variation, the fractional loss per cycle is

$$(\pi \mathrm{x} \cos \phi)^{-1} \int_0^{2\pi} \left| \left[\frac{1+\mathrm{x}\sin\theta}{\delta\mathrm{x}+1} - \ln\left(\frac{1+\mathrm{x}\sin\theta}{\delta\mathrm{x}+1} \right) - 1 \right] \sin(\theta - \phi) \right| \mathrm{d}\theta$$

where x is the ratio of the temperature oscillation amplitude to the average temperature, δ is ratio of the change in the mean temperature at the transition to the temperature oscillation amplitude, and ϕ is the phase angle between the mass flow and the temperature oscillation.

Approximate solutions are also developed for cases relevant to pulse tube and Stirling cryocoolers.

INTRODUCTION

Enthalpy flow analysis has become a useful tool in understanding how regenerative cryocoolers work.[1] Finite enthalpy flows occur in adiabatic regions of coolers and in regions with poor heat transfer between the working fluid and the surroundings. Regions with finite enthalpy flow include the pulse tube in pulse tube coolers, the expansion space in Stirling and G-M coolers that are operating to fast for isothermal expansion, and between a valveless compressor and the aftercooler.

Enthalpy flow analysis has mostly been used to analyze the gross cooling power. However this technique may also be used to quantify losses within coolers. This paper discusses a loss that

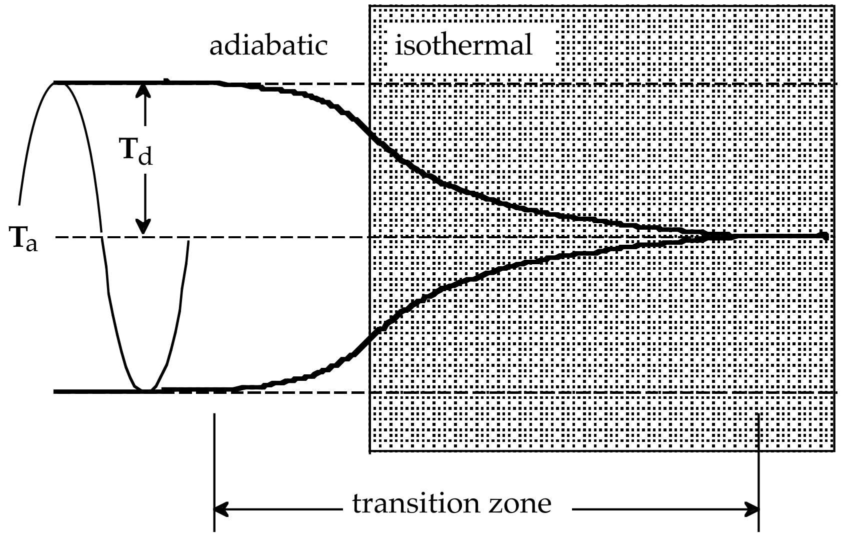

Figure 1. Representation of the change in the amplitude of the temperature oscillation at a transition from a finite enthalpy flow to isothermal flow. The amplitude changes from T_d to 0. In this example the mean temperature T_a does not change.

occurs when there is a change in enthalpy flow, in particular, when the enthalpy flow changes between a finite value and zero. This is shown pictorially in Figure 1. Such transitions may be found at the junction of heat exchangers and adiabatic sections.

In regions with enthalpy flow the local temperature, T, and mass flow, $\dot{m}$, of the working fluid oscillate. The enthalpy flow per cycle is given by

$$H_d = C_p \int_0^{2\pi/\omega} \dot{m} T \, dt \tag{1}$$

On entering an isothermal section, the enthalpy flow goes to zero, because the amplitude, T_d, of the temperature oscillation goes to zero. The transition occurs over a finite distance in a transition zone. In this zone gas elements undergo a temperature change from T to T_a, the mean temperature. The time it takes for this transition is small compared to the cycle time ($2\pi/\omega$). This is the same as saying that the displacement of a gas element during a cycle is large compared to to the length of the transition zone. Thus, we may treat the process as quasi-static. During the time it takes for an element of gas to cross the transition zone, $\dot{m}$ and the pressure, P, do not change and T in the adiabatic section does not change. However, while crossing the transition zone, the temperature of the gas element changes. The temperature change is not caused work being done because, on the time scale of crossing the transition zone, P and $\dot{m}$ are constant. Thus, the temperature change is caused by transferring heat to or from the surroundings. The rate heat is transferred by an incremental change in temperature, $d\theta$, of the gas element is

$$d\dot{Q} = C_p \dot{m} \, d\theta \tag{2}$$

(Positive heat flow is from the heat exchanger to the gas.) The total heat transferred by an element of gas is

$$dQ = C_p \dot{m} \left(\int_{T}^{T_a} d\theta \right) dt \tag{3}$$

The net heat transfer per cycle is found by integrating over one cycle:

$$Q = -C_p \int_0^{2\pi/\omega} \dot{m} \left(T - T_a \right) dt = \Delta H_d \tag{4}$$

This is the expected result since for infinitesimal processes dH = dQ + V dP and dP = 0 for this process.

However the process is not reversible. During the incremental process described above the heat flows across a temperature difference. Thus entropy is produced. The rate entropy changes is

$$d\dot{S} = C_p \left| \dot{m} \left(\frac{1}{T_a} - \frac{1}{\theta} \right) d\theta \right| \tag{5}$$

where θ is the instantaneous temperature of an element of gas within the transition zone and the absolute value sign is a reflection of the second law of thermodynamics, that dS ≥ 0 for all processes. The entropy change within the transition zone is found by integrating eq. (5) over the complete temperature change:

$$dS = C_p \left| \dot{m} \left(\int_{T_a}^{T} \left(\frac{1}{T_a} - \frac{1}{\theta} \right) d\theta \right) \right| dt \tag{6}$$

In a more generalized case the mean temperature is different in the two regions. This is depicted in figure 2 where the mean temperature changes from T_a in the adiabatic region to $(T_a + T_b)$ in the isothermal region. This situation is representative of the expander in regenerative coolers and of the flow between a compressor and its aftercooler. Equation (6) may be easily rewritten for this situation:

$$dS = C_p \left| \dot{m} \left(\int_{T_a + T_b}^{T} \left(\frac{1}{T_a + T_b} - \frac{1}{\theta} \right) d\theta \right) \right| dt \tag{7}$$

Similarly eq. (4) may be rewritten as

$$Q = -C_p \int_0^{2\pi/\omega} \dot{m} \left[T - \left(T_a + T_b \right) \right] dt = \Delta H_d \tag{8}$$

Equation (7) may be integrated over a cycle to yield the entropy produced per cycle:

$$\Delta S = C_p \int_0^{2\pi/\omega} \left| \dot{m} \left(\int_{T_a + T_b}^{T} \left(\frac{1}{T_a + T_b} - \frac{1}{\theta} \right) d\theta \right) \right| dt \tag{9}$$

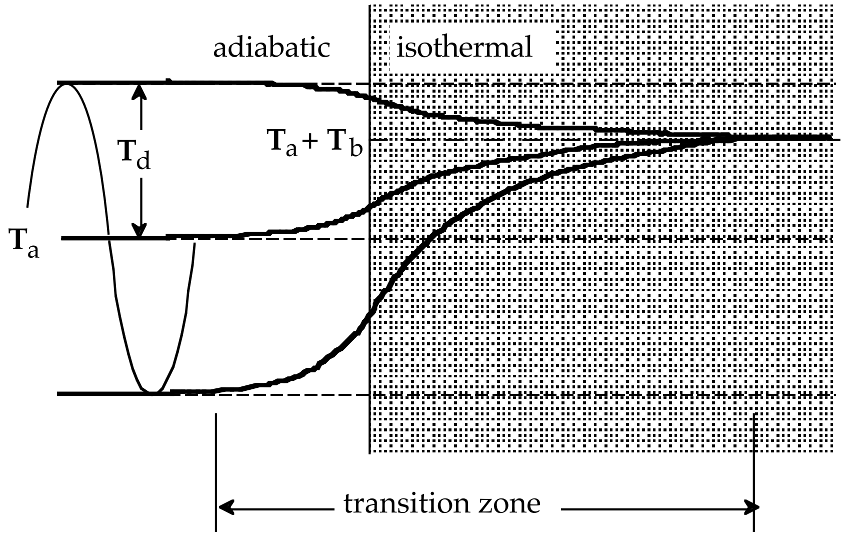

Figure 2. Representation of the change in temperature amplitude at a generalized transition from a finite enthalpy flow to isothermal flow. The amplitude changes from T_d to 0. The mean temperature changes from T_a in the adiabatic region to $(T_a + T_b)$ in the isothermal region.

This entropy is produced by an irreversible process, the flow of heat across a temperature gradient. The enthalpy change during a process can be expressed as dH = T dS + V dP. Here, there is no pressure loss. Thus, the irreversible entropy reduces the enthalpy flow available for refrigeration. I.e., a portion of the heat transfer, eq. (8), is the result of this entropy production and is not available for cooling an external load. The lost enthalpy is $T_a \Delta S$. The fractional enthalpy lost per cycle is

$$\varepsilon = T_a \Delta S / H_d \qquad (10)$$

For simplicity we will assume that the temperature and mass flow are sinusoidal:

$$T = T_a + T_d \sin(\omega t) \qquad (11)$$

$$\dot{m} = \dot{m}_d \sin(\omega t + \phi) \qquad (12)$$

where the subscripts a and d refer to the mean and dynamic components, ω is the frequency and ϕ is the phase shift. Substituting eqs. (8), (9), (11), and (12) into eq. (10) and integrating over θ results in :

$$\varepsilon = (\pi x \cos\phi)^{-1} \int_0^{2\pi} \left| \left[\frac{1 + x\sin\theta}{\beta x + 1} - \ln\left(\frac{1 + x\sin\theta}{\beta x + 1} \right) - 1 \right] \sin(\theta + \phi) \right| d\theta \qquad (13)$$

where $x = T_d/T_a$, $\beta = T_b/T_d$, and $\theta = \omega t$. The variables x, β, and ϕ are constrained to the following ranges by physical considerations: $0 \le x \le 1$, $-1 \le \beta x$, and $-\pi/2 < \phi < \pi/2$.

(The loss, eq. (10), uses T_a as the reference temperature. This temperature was chosen because it is the mean temperature of the gas and it is the gas that carries the enthalpy flow. However, in cooler analysis it is more common to use the temperature of the isothermal sections, (T_a + T_b), as the reference temperature. In this case one can define a loss: $\square^{\square} = (T_a + T_b)\square S / H_d$ where $\square^{\square} = \square(T_a + T_b)/T_a = \square(\square x + 1)$. This paper will deal with □ rather than □*.)

APPROXIMATE SOLUTION

The integrand of eq. (13) is shown in figure 3. This is not a simple function to integrate in closed form. Fortunately, there are ways of approximating it. It is also possible to numerically integrate eq. (13).

A lower limit, $\square_\ell$, to eq. (13) can be found by making use of the relation: $\int |I| d\square \geq \left| \int I\, d\square \right|$. These two integrals are equal when I does not change sign over the range of integration; i.e., $I \geq 0$ or $I \leq 0$ throughout the integration range. Here

$$I = \left[\frac{1 + x\, sin\square}{\square x + 1} \square\, ln\left(\frac{1 + x\, sin\square}{\square x + 1} \right) \square 1 \right] sin(\square\square\square) \qquad (14)$$

Thus,

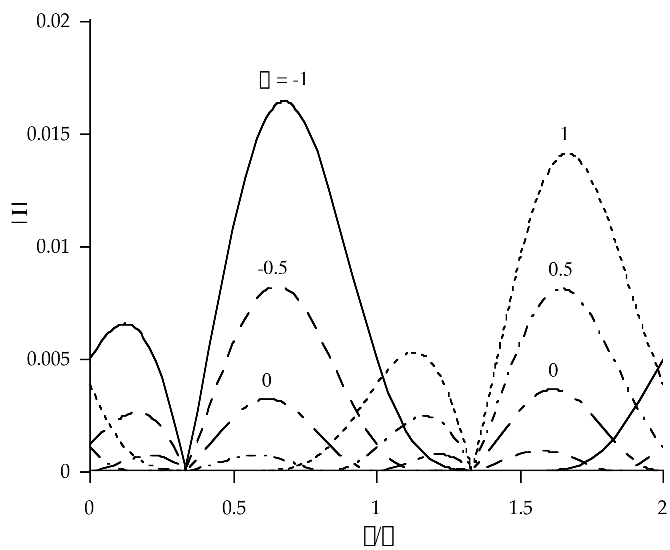

Figure 3. Plot of $|I(x, \square, \square, \square)|$, where I is given by eq. (14), for x = 0.1, □=□/3, and five values of □ between -1 and 1.

$$\square \ge \square_\ell = (\square x \cos\square)^{\square 1} \left| \int_0^{2\square} \left[\frac{1 + x \sin\square}{\square x + 1} \square \ln\left(\frac{1+x\sin\square}{\square x+1}\right) \square 1 \right] \sin(\square\square\square)\, d\square \right| \quad (15)$$

An approximate form of $\square_\ell$ can be found by expanding the logarithmic term in eq. (15) as a power series in x and keeping the lowest order term. This results in

$$\begin{aligned} \square_\ell \;\square\; \square\square &= |\square|\, x\, (\square x + 1)^{\square 1} \\ &= |T_b| (T_a + T_b)^{\square 1} \end{aligned} \quad (16)$$

This approximation is the same as saying that the heat transfer process nominally involves the irreversible transport of heat from T_a to $T_a + T_b$; which generates entropy: $S = Q\left| T_a^{\square 1} \;\square\; (T_a + T_b)^{\square 1} \right|$. The same result can be reached by approximating the 1/□ term in eq. (7) by $1/T_a$. However eq. (16) is derived, it is only an approximate lower limit. It does not have the □ dependence that is explicitly contained in eq. (13). Nor does it adequately predict the behavior near □ = 0. Equation (7) and, therefore, eq. (13) are clearly not 0 at □ = 0, yet eq. (16) is.

Another approach is to find an approximate closed form solution of eq. (13) in the limit of small x and □. The integral in eq. (13) can be written as $\int |I|\, d\square$. This integral may be rewritten, since $(\square x+1) \ge 0$, as $\int |I|\, d\square = (\square x+1)^{-1} \int |J|\, d\square$ where

$$J = (\square x + 1)\, I = \left[\square x \sin\square \;\square\; \square\square x \;\square\; (\square x + 1) \ln\left(\frac{1 + x\sin\square}{\square x + 1}\right)\right] \sin(\square\square\square) \quad (17)$$

This can be expanded in a Taylor series of x. Keeping the lowest order term results in

$$J \;\square\; \tfrac{1}{2} (\square \;\square\; \sin\square)^2\, x^2 \sin(\square\square\square) \quad (18)$$

The function $|J|$ may be integrated by first dividing it into segments where $J \ge 0$ or $J \le 0$ and then integrating each segment separately. These regions are illustrated in figure 4. The boundaries between the $J \ge 0$ and $J \le 0$ segments only depend on □ and □.

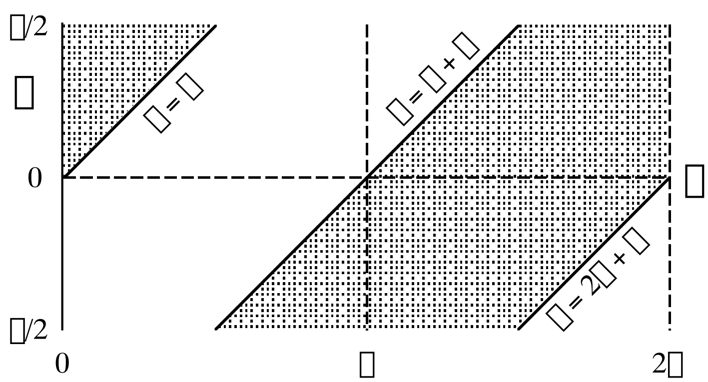

Figure 4. Regions in (□,□) space where eq. (18) is positive (unshaded) or negative (shaded).

Integrating $\left|J(x,\alpha,\phi,\theta)\right|$ over θ in this piecewise fashion yields

$$\eta''(\alpha,\phi) = \left(\tfrac{1}{3}\cos(2\phi) + 2\alpha^2 + 1\right)\frac{x}{\pi(\pi x + 1)\cos\phi} \qquad (18)$$

COMPARISON TO NUMERICAL SOLUTION

It is possible to solve eq. (13) by numerical integration. Figure 5 shows such a solution for x = 0.1. The solution has a broad central region (where ϕ and α are small) in which η is slowly varying. At ($\phi = 0$, $\alpha = 0$) η has a value of about $4x/3\pi$ for a loss of about 4.2% when x = 0.1. Outside of this region, η increases rapidly. Figures 6, 7, and 8 show comparisons between η and the approximations, η' and η''. These comparisons are done around several operating conditions that are representative of conditions that might occur in coolers: $\phi = 0$, x = 0.1, and α = -1, 0, and 1.

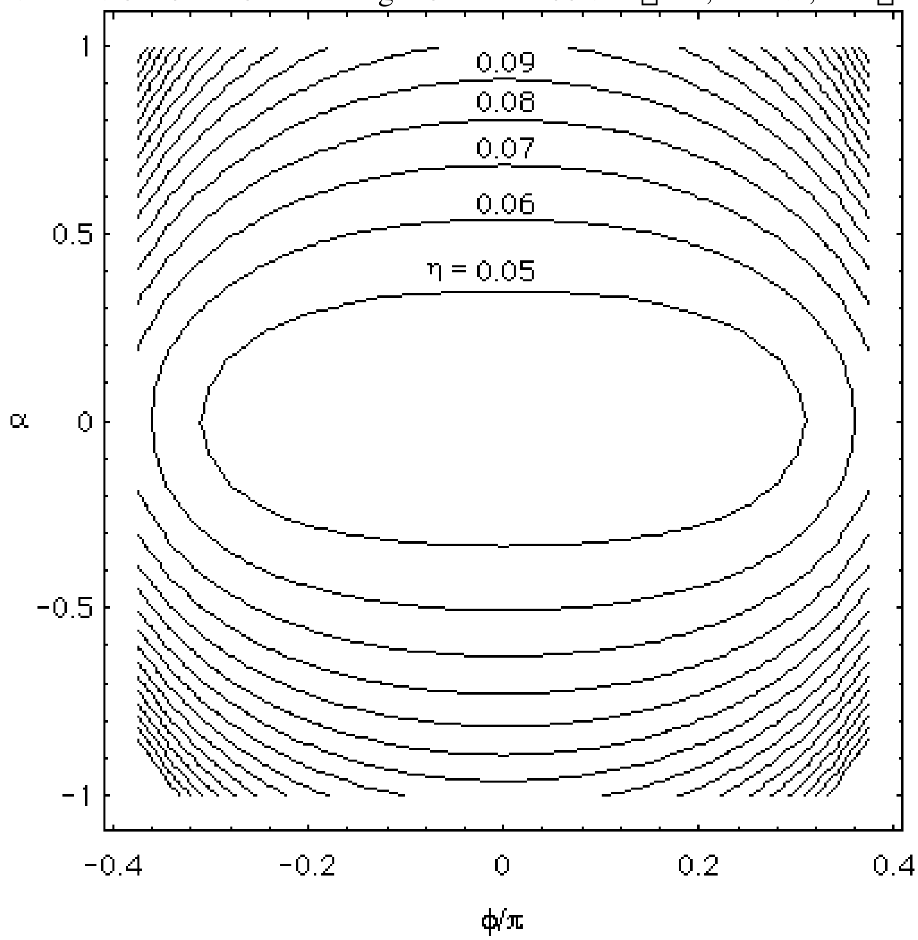

Figure 5. Contours of constant loss, η, in (ϕ,α) space for x = 0.1. The contours are 0.01 apart.

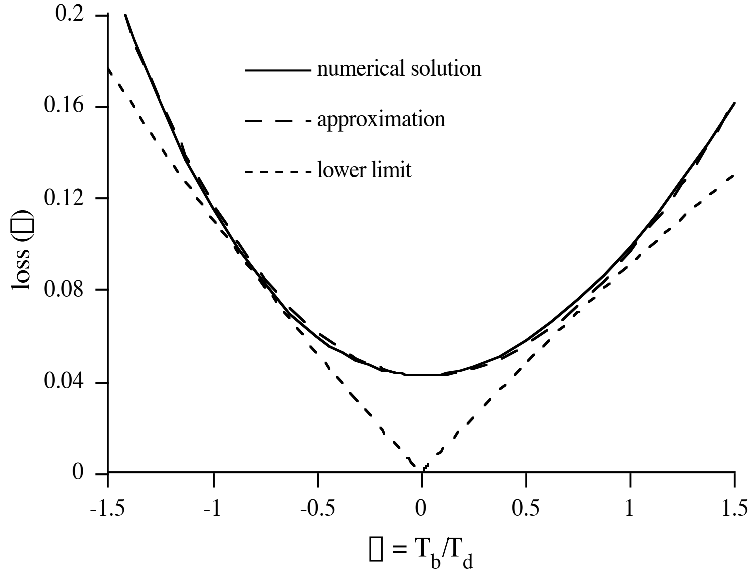

Figure 6. Comparison as a function of □ of the approximate lower limit, □', and the approximation, □", with the numerical solution of □ for x = 0.1 and □ = 0.

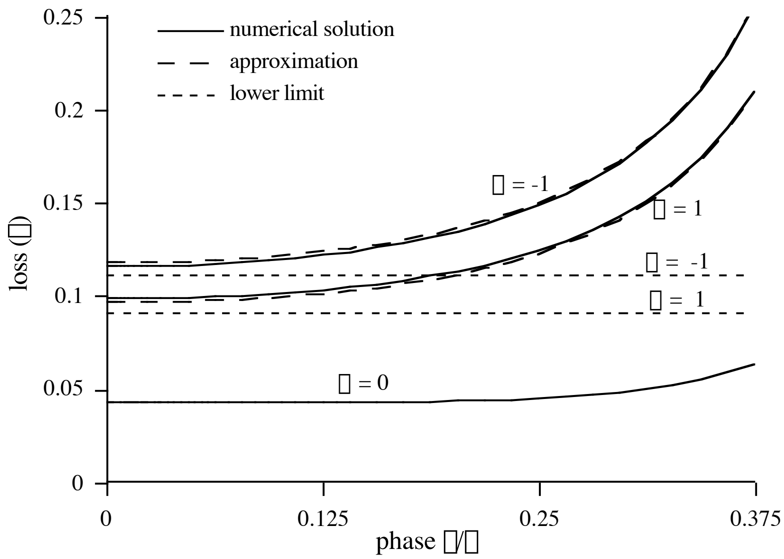

Figure 7. Comparison as a function of phase, □, of the approximate lower limit, □', and the approximation, □", with the numerical solution of □ for x = 0.1 and □ = -1, 0, and 1. For □ = 0, □ and □" are indistinguishable on this scale and □' = 0.

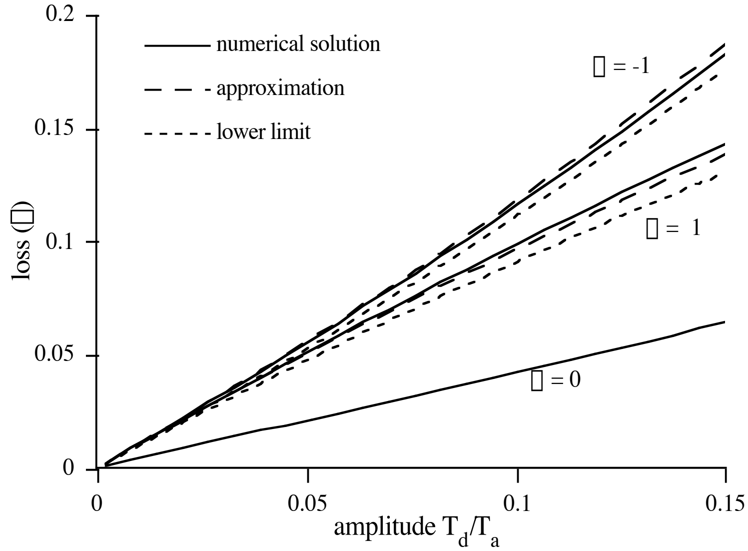

Figure 8. Comparison as a function of amplitude, $x = T_d/T_a$, of the approximate lower limit, □', and the approximation, □", with the numerical solution of □ for □ = 0 and □ = -1, 0, and 1. For □ = 0, □ and □" are indistinguishable on this scale and □' = 0.

The approximate lower limit, □', behaves as was expected from the earlier discussion. It is not a good approximation. It does not have a phase dependence nor does it adequately predict the behavior for small □ or for large □. It is only a good predictor for □ ~ 0 and 0.5 ≲ |□| ≲ 1. The approximation □" is a far better predictor. It tracks the □ and x dependence of □ over the range of interest. The errors are insignificant.

DISCUSSION

Table 1 gives values of x and □ that might be expected for coolers. For hot heat exchangers, the exchanger temperature is near the minimum gas temperature. For cold heat exchangers, the exchanger temperature is near the peak gas temperature. The total loss may be found by combining the individual losses using: $\square = 1 \square \square_i \left(1 \square \square_i\right)$. For the case given in Table 1 the overall loss would be 30%. This may be compared to the model of Wang et. al.[2] A 30% loss in gross cooling power almost completely accounts for the difference between the calculated and experimental results they present. Such good agreement may be just coincidental. The losses calculated in Table 1 assume sinusoidal waveforms; the real waveforms are not. An integration of eqs. (8) and (9) using the real waveforms would give a different result. Also, the values listed in Table 1 are only rough estimates.

Table 1: Typical values

location	x	□	□	□
compressor/aftercooler	0.1	-1	0	0.12
cold heat exchanger *	0.1	1	0	0.10
hot heat exchanger *	0.1	-1	0	0.12

∗ estimated from ref. 2

The data presented in the table are consistent with the published data but may not adequately describe the process. Furthermore, some of the enthalpy flow losses may be already included in their model. As will be discussed below, the loss at the hot heat exchanger should not be included in the overall loss. Excluding this would reduce the total loss from 30% to 20%.

In an ideal pulse tube, all of the enthalpy flow must be irreversibly dissipated at the hot heat exchanger.[3] The mechanism discussed in this paper can account for part of this dissipation. (Another source of loss at the hot end is the V dP work done in the orifice.) Since this total dissipation is inherent to pulse tubes, most models probably already account for it. In any case a loss in enthalpy flow at the hot end of a pulse tube does not reduce the cooling power.

The estimate of □ found in Table 1 was based on calculated waveforms at the ends of the pulse tube. These waveforms are probably taken from within the transition zone. Therefore Table 1 probably under estimates the loss. To improve on this, we can take a look at the temperature profile in the pulse tube. This profile has been measured for a pulse tube.[1] Estimated parameters based on these measurements are summarized in Table 2

There is a difficulty in using this data in the enthalpy flow loss model. The model assumes that T_a is constant in the adiabatic region. Yet, in a real pulse tube T_a has a steep gradient. This is especially true near the hot end. We would like to use temperature estimates that are neither affected by the transition zone nor by this gradient. One method is to use the minimum (or maximum) value of T_a. This choice reduces the effect of the transition zone. However, since the minimum (or maximum) occurs away from the end, the estimate is still affected by the gradient. This results in underestimating □. Another choice is to extrapolate T_a from the central region near the minimum (or maximum) to the ends of the pulse tube. Both of these methods are used in Table 2.

The two methods give about the same result (□ ~12%) for the cold end loss. This loss is similar to the one found in Table 1.

Table 2: Pulse Tube Performance Parameters ‡

position	T_d	T_a	$T_a + T_b$	□
cold end *	10 K	185 K	200 K	0.09
†	9.7 K	180 K	200 K	0.15
hot end §	30 K	360 K	300 K	0.31
†	33-35 K	400-425 K	300 K	0.75-1.11

‡ estimated from ref. 1

* using minimum reported T_a

§ using maximum reported T_a

† T_a estimated by extrapolating to end of pulse tube
T_d estimated by assuming T_d/T_a is same for both cases

The hot end loss is quite different. Using the extrapolated data, the loss is ~90%. Thus, this accounts for a substantial portion of the required dissipation. It also accounts for the large overshoot of the gas temperature above the hot end heat exchanger, a trait of pulse tubes. Again, this good agreement between calculated and expected loss may just be fortuitous. At the hot end of a pulse tube the mass flow is small. Only a small fraction of the mass flow through at the regenerator makes it through the orifice at the hot end. The assumption that the time to cross the transition zone is << 2□/□ is questionable. A gas element may not even traverse the transition zone at the hot end. It may only move through part of the zone during a cycle. Thus, some of the temperature change may due to changes in $\dot{m}$ and P. Such changes are reversible and do not increase the entropy. Therefore, the loss calculated here is an upper limit for the mechanism discussed in this paper.

SUMMARY

A method of calculating the losses caused by transitions in enthalpy flow has been developed in the convective limit. In this limit, the time an element of gas takes to cross the transition zone is << 2□/□. Equivalently, the displacement of a gas element during a cycle is >> the length of the transition zone. Since, these conditions are not always met in a real cooler; the loss presented here is an upper limit of the real loss from this mechanism. Additional losses from other mechanisms may also be present.

The losses caused by transitions in enthalpy flow are significant. Losses ~10% are typical for aftercoolers and cold heat exchangers in regenerative cryocoolers. Even greater losses can be expected at the hot end of the pulse tube where this transition loss accounts for a large part of the heat rejected. While the examples used here were for pulse tubes, other regenerative coolers have similar values of x and □ at their cold heat exchangers and in their compressors. Thus, they should have similar losses. For real coolers there is likely to be a significant deviation from these calculated results which comes from the real waveforms not being sinusoidal.[1,2]

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